Demonstration in Euclidean Geometry

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Abstract

Drawing out a model or a generalization in mathematic classes for cycle 3 – Grades 7,8 and 9 – is one of the major difficulties the learners face. However, exposing the learners to excessive practice and training on strategies and demonstrations through analysis and intermediate hints can lead to appreciable improvement in the learners' abilities to solve mathematical problems which come to closure by carrying out a generalization. The aforementioned hypothesis was justified through conducting two examples on particular segments in a triangle in Grades 7, 8, and 9. Two theories support the approach of this action research. The theory of the Dutch researcher - Pierre Marie Van Hiele - who divided the Geometry learning into 5 sequential or linear steps: visualization, analysis, informal deduction, deduction, and rigor. On the other hand, the theory of the French researcher - Alain Kuzniak - who presented the Geometry learning as a back and forth navigation between three levels of Geometry specifying in each level the role of intuition, experience, deduction, kinds of spaces, status of drawing, and the privilege aspect.

Keywords: Geometrical Demonstration, Mathematical Deduction, Mathematical Modelling, Mathematical Reasoning, Teaching – Learning Geometry.

Introduction

Our students in cycle 3 face difficulties in solving Math problems. Subdividing the question into smaller parts or giving hints…lead to facilitating the problems and therefore the student is capable of solving such problems.

Many studies conducted in different countries and on different levels of Geometry teaching whether intermediate, secondary, and even college, all proved the need for a change in the curriculum and in the classroom so that students can upgrade their thinking levels to informal and formal deduction when studying geometry.

One of the studies:"Developing geometrical reasoning in the secondary school: outcomes of trialing teaching activities in classrooms". A Report from the Southampton/Hampshire Group to the Qualifications and Curriculum Authority - Margaret Brown, Keith Jones & Ron Taylor - November 2003
(This study is conducted on secondary students).

Another example of the conducted studies:"The Impacts Of Undergraduate Mathematics Courses On College Students' Geometric Reasoning Stages" - *Nuh Aydin1 - Kenyon College, Ohio& Erdogan Halat2 - Afyon Kocatepe University, Turkey.*(Conducted on 149 college students, and provides detailed history of studies done on learning geometry.)

An Action Research is done to suggest an optimistic solution for intermediate grade levels' students - specifically students of Grades 7, 8 and 9 - and teachers to approach Geometry learning whether it is related to modelling or to demonstration and generalization.

Definition of Action Research:

> "Action research is a natural part of teaching. Teachers are continually observing students, collecting data and changing practices to **improve student learning** and the classroom and school environment. Action research provides a framework that guides the energies of teachers toward a better understanding of why, when, and how students become better learners." A. Christine Miller (2007)

According to Miller, there are five phases of action research:

1. Selecting an area or focus:

Solving Math problems in cycle 3 (Grade levels: G7, G8, and G9)
Two Examples are considered under the same title: "Particular segments in a triangle"

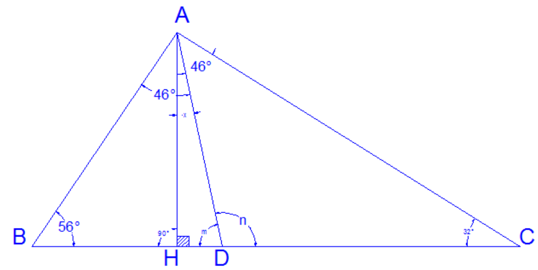
Problem One: Geometry is an obligatory passage from Arithmetic to Algebra

Prerequisites:

Properties in a Triangle as:

- The sum of angles inside the triangle is 180◦
- The notion of exterior angles.
- Bisectors, medians, and heights in any triangle.

a) Numerical Approach:

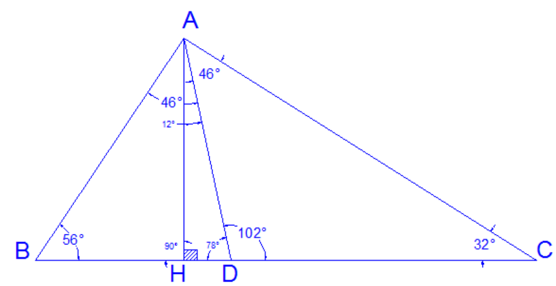


Given:

Any triangle ABC, AC > AB
[AD] is the bisector of angle A in triangle ABC
[AH] is a height in triangle ABC
$\hat{n} > \hat{m}$;

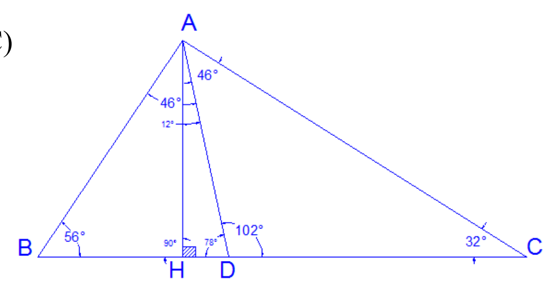
Required:

Determine $(\hat{n} - \hat{m})$
Calculate $\hat{x}$ the angle between the height and the bisector through vertex A.



First Solution:

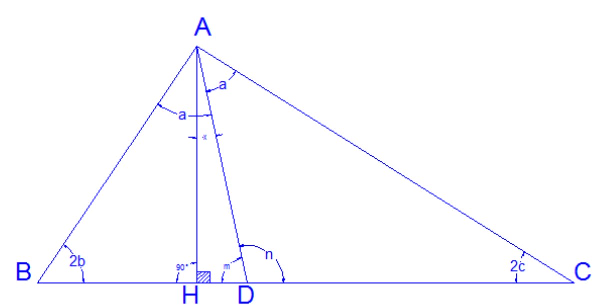
$\hat{A}+\hat{B}+\hat{C}$ = 92+56+32 = 180◦ (sum of angles in a triangle)
$\hat{n}$= 56+46 = 102◦ (exterior angle to triangle ABD)
$\hat{m}$ = 32+46 = 78◦ (exterior angle to triangle ADC)
$\hat{n} - \hat{m}$ = 102-78 = 24◦
$\hat{x}+\hat{m}$+90 =180◦ (sum of angles in triangle AHD), then
$\hat{x}$= 90-$\hat{m}$ = 90-78 = 12◦ then $\hat{x} = \frac{24°}{2}$ =12◦



Second Solution:

$\hat{A}+\hat{B}+\hat{C}$ = 92+56+32 = 180◦ (sum of angles in triangle ABC)
$\hat{n} + 46 + 32$ = 180◦ (sum of angles in triangle ADC)
$\hat{m}$ +46+56 = 180◦ (sum of angles in triangle ABD)
$\hat{n} + 46 + 32 = \hat{m}$ +46+56 , then
$\hat{n} - \hat{m}$ = 46+ 56 - 46 -32 = 24◦
$\hat{x}+\hat{m}$+90 =180◦ (sum of angles in triangle AHD), then
$\hat{x}$= 90-$\hat{m}$ = 90-78 = 12◦ then $\hat{x} = \frac{24°}{2}$ =12◦

b) Modelling through literacy calculation



$\hat{n}=\hat{a}+2\hat{b}$ (exterior angle to triangle ABD);
$\hat{m}=\hat{a}+2\hat{c}$ (exterior angle to triangle ACD);
Then, $\hat{n}$-$\hat{m}$= $\hat{a}$ +2$\hat{b}$ - ($\hat{a}$+2$\hat{c}$) = 2$\hat{b}$ - 2$\hat{c}$
$\hat{x}$=90-$\hat{m}$ (in triangle AHD);
$\hat{n}=\hat{x}$+90 (exterior angle to triangle AHD), then 90=$\hat{n}$-$\hat{x}$
$\hat{x}$= ($\hat{n}$-$\hat{x}$) - $\hat{m}$ then
2$\hat{x}$= $\hat{n}$-$\hat{m}$=2$\hat{b}$-2$\hat{c}$; therefore,
$\hat{x}=\hat{b} - \hat{c}$ or $\hat{x} = \frac{1}{2}(2\hat{b} - 2\hat{c}) = \frac{1}{2}(\hat{n} - \hat{m})$

The Model: In any triangle ABC (of corresponding angles $\hat{A}$, $\hat{B}$, and $\hat{C}$) such that AC > AB, the angle between the bisector and the height at vertex A or angle $\hat{A}$ is always equal to $\frac{1}{2}(\hat{B}$-$\hat{C})$.

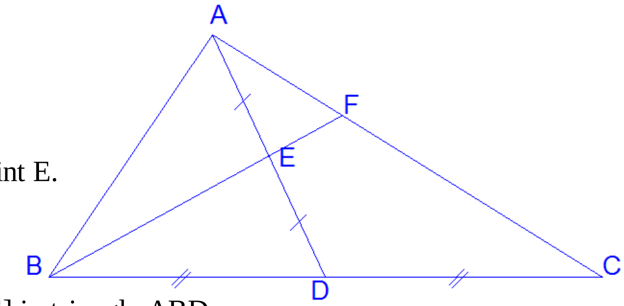
Problem Two: Geometric Demonstration - Problem analysis and Generalization

Prerequisites:

- The midpoint theorem in Geometry.
- The median in a triangle

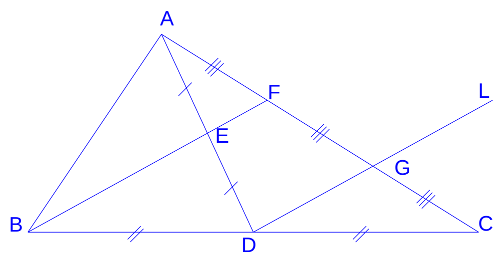
Given:

Any Triangle ABC
[AD] is a median in triangle ABC
Draw a semi-line through B and extend it to intersect side [AC] in point F such that [BF] intersects[AD] in its midpoint E.
Show that AF = $\frac{1}{3}$ AC

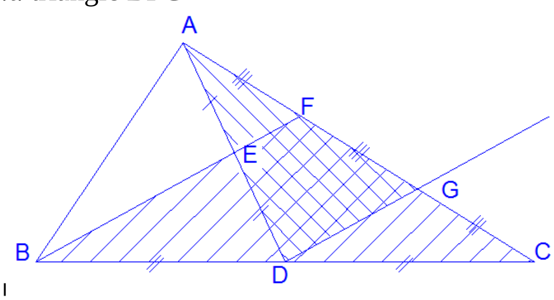


Hints:

1- Reproduce the given figure and define the nature of [BE] in triangle ABD.
2- There is a need to divide [AC] into 3 equal parts.



3- Draw the semi line [DL) parallel to (BF); [DL) intersects side [AC] in point G.
4- Define the position of G w.r.t. triangle BFC



5- Consider the two triangles: ADG and BFC
6- Apply the midpoint theorem in triangles ADG and BFC

Solution:

Apply Midpoint Theorem:

Triangle BFC: $\frac{CD}{CB} = \frac{CG}{CF} = \frac{1}{2}$ Triangle ADG: $\frac{AE}{AD} = \frac{AF}{AG} = \frac{1}{2}$

Then AF = FG = GC or AF = $\frac{1}{3}$AC

2. Collecting data

	Table of Results	Example 1	
Exercise 1	Grade 7 percentage of correct answers	Grade 8 percentage of correct answers	Grade 9 percentage of correct answers
Numerical Application: n-m = numerical value	75%	98%	98%
Literal Calculation: Verify that: n-m = 2b - 2c	15%	35%	72%
Modelling: Deduce that:x = 1/2(2b-2c)	0%	12%	30%

	Table of Results	Example 2	
6 Hints were given:	Grade 7 percentage of correct answers	Grade 8 percentage of correct answers	Grade 9 percentage of correct answers
1- Define nature of [BE] in triangle ABD 2- The need to divide [AC] into 3 equal parts After Hints 1 and 2	10%		
3- Draw [DL) parallel to [BF] 4- Identify position of G w.r.t triangle BFC After Hints 1 to 4		27%	
5- Consider the triangles ADG and BFC 6- Apply the midpoint theorem After Hints 1 to 6			45%

3. Organizing data:

Two inferences from the collected percentages:

- It is difficult to pass from arithmetic to algebra, and Geometry is a need
- Most students fail to solve math problems without hints.

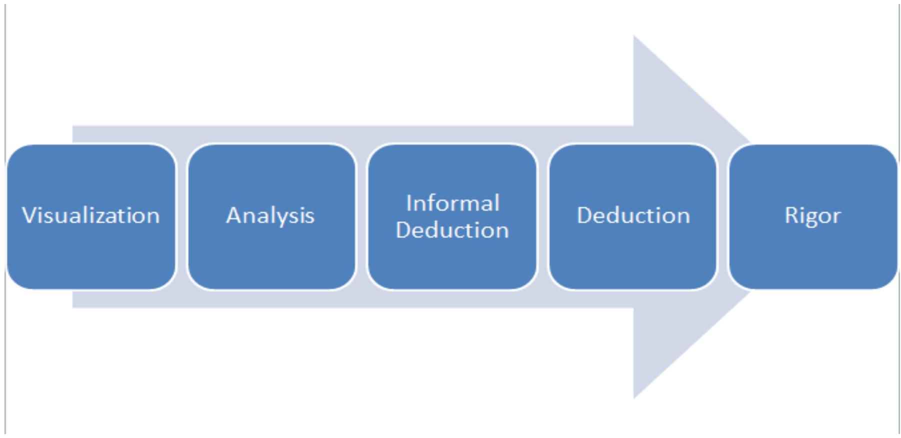
4. Analyzing and interpreting data-Studying the professional literature:

Two supporting theories to our approach:

- **Pierre M. Van Heile (Dutch)**
- **Alain Kuzniak (French)**

Pierre M. Van Hiele

Model of the Development of the Geometric Growth



According to M. Crowley, in "The van Hiele Model of the Development of Geometric Thought."

1- Visualization:Geometric concepts with respect to learner are holistic shapes without details or rules.

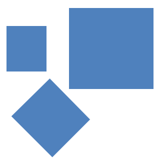


Figure a

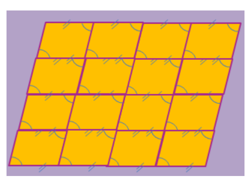
Figure b

The learner's achievements at this level cover related vocabulary, identifying different geometric shapes, can reproduce a given figure.

Van Hiele speaks of "spatial thinking" in this level (Van Hiele 1986).

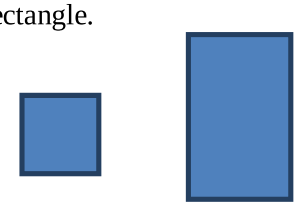
According to Van Hiele, the learner at this stage can differentiate between squares as in figure (a) and rectangles as in figure (b) because he had seen similar squares and rectangles but the learner cannot understand the properties of each like right angles or parallel opposite sides…

2- Analysis: "At this level, analysis of geometric concepts begins." Figures have parts or elements and not just a whole or form.



Interrelationships of properties within a geometric figure can be established by the learner at this level; for example, the learner can recognize that a parallelogram of parallel opposite sides mandates the equality between opposite angles.

3- Informal Deduction: At this level, learners can establish the interrelationships of properties both within one figure and among different figures; for example the learner can recognize that a square is a rectangle as it admits all the properties of a rectangle.



4- Deduction: At this level, the learner can use deduction to establish certain geometric theory in an axiomatic system. There is an understanding of the difference, relations between, and roles of terms, axioms, postulates, definitions, theorems, and proof. At this stage there is a possibility for learner to develop a proof and in more than one way, understand the interaction between necessary and sufficient conditions, and to distinguish between a statement and its converse.**(Van Hiele speaks of "logical mathematical thinking".)**

5- Rigor: At this stage the learner can work in different axiomatic systems, such as non-Euclidean geometries and can compare between different systems. Geometry is lifted to its abstract level.

Properties of theVan Hiele Model:

1. Sequential

It is a linear procedure where the learner follows a certain order.

2. Advancement

Van Hiele points out that it is possible to teach "a skillful pupil abilities above his actual level, like one can train young children in the arithmetic of fractions without telling them what fractions mean, or older children in differentiating and integrating though they do not know what differential quotients and integrals are" (Freudenthal 1973, p. 25).

3. Intrinsic and Extrinsic

The inherent objects at one level become the objects of study at the next level.

4. Linguistics

"Each level has its own linguistic symbols and its own systems of relations connecting these symbols" (P.M. Van Hiele 1984a, p. 246).

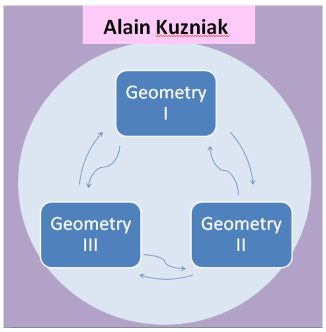
5. Mismatch:

If the learner is at one level and instruction is at a different level, the desired knowledge acquiring may not be accomplished. In particular, if the instructor, instructional materials, content, vocabulary, and so on… are at a higher level than the learner, the learning and progress may not happen.

Kuznizk opinion in Van Hiele approach:

"For us, this way to Geometry is to a great extent correct but too strictly linear and univocal especially if we want to understand the obstacles met by adults who want to become teachers." (Houdement, C & Kuzniak, A, 2003, p1)

"We freely use Van Hiele's levels outside his theory to give us good benchmarks about the levels of the mathematical thinking of the students." (Houdement, C & Kuzniak, A, 2003, p6)



Alain Kuzniak, as presented in "Elementary Geometry Split Into Different Geometrical Paradigms" in precedings to cerme3, (2003) classifies Geometry learning into three levels as follows:

Geometry I (Natural Geometry).

In this level of Geometry the reasoning is based by nature to experience and intuition; it is based on reality. At this level the learner operates according to his immediate perception then experiments and deduces after acting on the material objects and the use of instruments.

Example: construct a triangle the length of its sides are 4cm, 8cm, and 10cm.

Students can construct the required triangle with sticks of the given lengths. Later on, the students can construct the triangle by drawing on a paper using the instruments as ruler and compass.

Geometry II (Natural Axiomatic Geometry).

At this level, the student can justify the validity of existence based on the hypothetical deductive laws in an axiomatic system. The necessary system of axioms must be as close as possible to intuition or reality around the learner.

Example: Certain triangles can look strange or even do not exist for a combination of lengths as 4cm, 4cm, and 10cm. Here arises the idea of axiom correlation between lengths of three sides for a triangle to exist

	Geometry I (Natural Geometry)	Geometry II (Natural Axiomatic Geometry)	Geometry III (Formalist Axiomatic Geometry)
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to the figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to schemas of the reality	Logical
Deduction	Near of the Real and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms.
Kind of spaces	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean Space
Status of the drawing	Object of study and of validation	Support of reasoning and "figural concept"	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-Evidence and construction	Properties et demonstration	Demonstration and links between the objects. Structure.

(Houdement, C. & Kuzniak, A. (2003) TG7_Houdement_cerme3, p5)

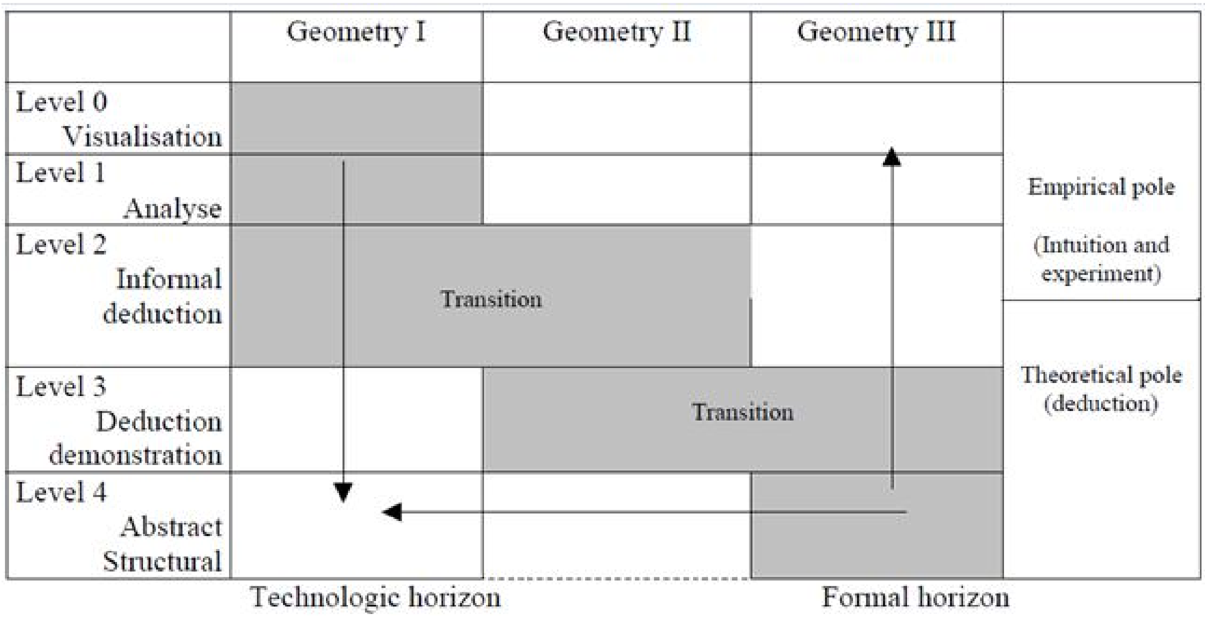
Geometry III (Formalist Axiomatic Geometry).

At this level there is disconnection between reality and axioms. The system of axioms can have no relation with reality.In this phase of Geometry Abstraction is reached by using similar reasoning to Geometry II but independent from validity or existence or applications in real life. The idea which governs is the absence of contradictions or consistency.

Example: the relation between lengths of sides for a triangle to exist can now be replaced by the vector relation of Chasles' theorem which applies not only to triangles.

5. Taking action

Kuzniak versus Van Hiele:



(Houdement, C. & Kuzniak, A. (2003) TG7_Houdement_cerme3, p7)

Interpretation of the table:

The table shows that:

- Geometry I includes: visualization, analysis, and informal deduction.
- Geometry II includes: the transition from informal deduction to the level of deduction; and
- Geometry III includes: the transition from the level of deduction to the abstract level or rigor.

The table applied on the first example:

- Students of cycle 3 all have the skills of the first 2 levels in Van Hiele's model visualization and analysis
- Students in grades 7 and 8 almost fail to achieve the levels of transition from informal deductive to deductive while grade 9 can still manage to 75%.
- Students in all grade levels of cycle 3 fail to accomplish a transition from deductive level to the rigor or abstract.

The table applied on the second example:

- Students of Grade 7 fail to achieve the natural Geometry level; students of Grade 8 succeed in the natural Geometry but drown before reaching the natural axiomatic Geometry; and grade 9 students who succeeded in the first two levels only 45% were able to reach the level of the formalist axiomatic Geometry.

Conclusion

Training and practice are highly recommended for students in cycle 3 to get used to the literal calculations and modelling at early stages of Math learning. Hints can facilitate solving problems in addition to more practice in this field. Action research is done to improve student learning in parallel with reflecting positive environment in the classroom. Continuous practice is a crucial factor to accomplish the competency of literal calculation, algebraic modelling and geometric demonstration.

References

Aydin, N & Kocatepe, A. (2009). The Impacts Of Undergraduate Mathematics Courses On College Students' Geometric Reasoning Stages. *The Montana Mathematics Enthusiast*, ISSN 1551-3440, Vol. 6, nos.1&2, pp.151- 164.

Brown, M., Jones, K. & Taylor, R. (2003). Developing geometrical reasoning in the secondary school: outcomes of trialling teaching activities in classrooms. A Report from the Southampton/Hampshire Group to the Qualifications and Curriculum Authority. Retrieved from: http://eprints.soton.ac.uk/9801/1/Brown_etc_develop_geom_reasoning_Soton_final.pdf

Brown, M., Jones, K. & Taylor, R. (2004). Developing Geometrical Reasoning. Retrieved from: http://www.merga.net.au/documents/RP132004.pdf

Ching Chou, S. & Shan Gao, X. Automated Geometry Reasoning and its Applications. The Wichita State University. Retrieved from:http://www.cs.duke.edu/AutoDedFD/contributions/wkshop/chou.gao.html

Crowley, M. The van Hiele Model of the Development of Geometric Thought. Retrieved from: http://www.cns-eoc.colostate.edu/docs/math/mathactivities/june2007/The%20van%20Hiele%20Model%20of%20the%20 Development%20of%20Geometric%20Thought.pdf

Deliyianni, E., Gagatsis, A., Monoyiou, A. Michael, P., Kalogirou, P. & Kuzniak A. (2011). Towards a comprehensive theoretical model of students' geometrical figure understanding and its relation with proof. And Secondary Students behavior in proof tasks: understanding and the influence of the geometrical figure. And Kuzniak, A. Geometric work at the end of compulsory education. *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education.* Retrieved from:http://www.mathematik.uni-dortmund.de/~erme/doc/cerme7/CERME7.pdf

Gutiérrez,A., Kuzniak, A. & Straesser, R.(2005.) Research on Geometrical Thinking. *Proceedings of Cerme4. and*Kuzniak, A. & Rauscher, J.C. (2005). On The Geometrical Thinking Of Pre-Service School Teachers. *Proceedings of Cerme4. Retrieved from:*http://www.mathematik.uni-dortmund.de/~erme/CERME4/CERME4_WG7.pdf

Houdement, C. & Kuzniak A.(2003). Elementary Geometry Split Into Different Geometrical Paradigms. *Proceedings of Cerme3. European Research In Mathematics Education III* – G7. Retrievedfrom: http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG7/TG7_Houdement_cerme3.pdf